

## A NOTE ON EULERIAN POLYNOMIALS OF HIGHER ORDER

JONGSUNG CHOI\*

ABSTRACT. In this paper we derive some identities on Eulerian polynomials of higher order from non-linear ordinary differential equations. We show that the generating functions of Eulerian polynomials are solutions of our non-linear ordinary differential equations.

### 1. Introduction

It is well known that the generating function  $F(t, x)$  of Euler polynomials  $E_n(x)$  is given by

$$(1.1) \quad F(t, x) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}.$$

In the special case,  $x = 0$ ,  $E_n(0) = E_n$  is the  $n$ -th Euler number. From (1.1), we note that

$$(1.2) \quad E_0 = 1, \quad (E + 1)^n + E_n = 0, \quad \text{if } n > 0,$$

with the usual convention of replacing  $E^n$  by  $E_n$ . The generating function  $F_u(t, x)$  of Eulerian polynomials  $H_n(x|u)$  are defined by

$$(1.3) \quad F_u(t, x) = \frac{1 - u}{e^t - u} e^{xt} = \sum_{n=0}^{\infty} H_n(x|u) \frac{t^n}{n!},$$

where  $u \in \mathbb{C}$  with  $u \neq 1$  (see [1,2]). In the special case,  $x = 0$ ,  $H_n(0|u) = H_n(u)$  is called the  $n$ -th Eulerian number (see [2]). Sometimes that is called the  $n$ -th Frobenius-Euler number (see [3-7]).

From (1.1) and (1.3), we note that  $H_n(x|-1) = E_n(x)$ .

---

Received November 28, 2012; Accepted January 11, 2013.

2010 Mathematics Subject Classification: Primary 11B68; Secondary 34A34.

Key words and phrases: Eulerian numbers and polynomials.

The present Reserch has been conducted by the Reserch Grant of Kwangwoon University in 2012.

By (1.3), we get

$$\begin{aligned}
 (1.4) \quad \frac{1-u}{e^t-u} e^{xt} &= \left( \sum_{l=0}^{\infty} H_l(u) \frac{t^l}{l!} \right) \left( \sum_{k=0}^{\infty} \frac{x^k t^k}{k!} \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} x^{n-l} H_l(u) \right) \frac{t^n}{n!}.
 \end{aligned}$$

From (1.3) and (1.4), we get

$$(1.5) \quad H_n(x|u) = \sum_{l=0}^n \binom{n}{l} x^{n-l} H_l(u) = (H(u) + x)^n$$

with the usual convention of replacing  $H^n(u)$  by  $H_n(u)$ .

By (1.5), we obtain

$$\begin{aligned}
 (1.6) \quad 1-u &= \frac{1-u}{e^t-u} e^t - \frac{1-u}{e^t-u} u \\
 &= \sum_{n=0}^{\infty} (H(u) + 1)^n \frac{t^n}{n!} - \sum_{n=0}^{\infty} u H_n(u) \frac{t^n}{n!}.
 \end{aligned}$$

Thus, we get the recurrence relation for  $H_n(u)$  as follows:

$$(1.7) \quad H_0(u) = 1, \quad H_n(1|u) - u H_n(u) = (1-u) \delta_{0,n},$$

where  $\delta_{n,k}$  is Kronecker symbol (see [1-7]).

For  $N \in \mathbb{N}$ , the  $n$ -th Eulerian polynomials  $H_n^{(N)}(x|u)$  of order  $N$  are defined by generating function as follows:

$$\begin{aligned}
 (1.8) \quad F_u^N(t, x) &= \underbrace{\left( \frac{1-u}{e^t-u} \right) \times \left( \frac{1-u}{e^t-u} \right) \times \cdots \times \left( \frac{1-u}{e^t-u} \right)}_{N\text{-times}} e^{xt} \\
 &= \sum_{n=0}^{\infty} H_n^{(N)}(x|u) \frac{t^n}{n!}.
 \end{aligned}$$

In the special case,  $x = 0$ ,  $H_n^{(N)}(0|u) = H_n^{(N)}(u)$  are called the  $n$ -th Eulerian numbers of order  $N$  (see [1-7]).

In [6], T. Kim introduced important ideas to obtain some new identities between the sums of products of Frobenius-Euler polynomials and Frobenius-Euler polynomials of higher order. The main idea is to construct non-linear ordinary differential equations with respect to  $t$  which are closely related to the generating functions of Frobenius-Euler polynomials.

In this paper, we consider non-linear ordinary differential equations with respect to  $u$  not  $t$ . The purpose of this paper is to give some new identities on the Eulerian polynomials of higher order by using the non-linear ordinary differential equations with respect to  $u$ .

### 2. Constuction of non-linear differential equations

We define that

$$(2.1) \quad \begin{aligned} F &= F(u) = \frac{1-u}{e^t-u} \quad \text{and} \\ F^N(t, x) &= \underbrace{F \times \cdots \times F}_{N\text{-times}} e^{xt} \quad \text{for } N \in \mathbb{N}. \end{aligned}$$

We note that  $F(t, x) = F_u(t, x) = F e^{xt}$ .  
By (2.1), we get

$$(2.2) \quad \begin{aligned} F^{(1)} &= \frac{dF}{du} = -\frac{1}{1-u} \frac{1-u}{e^t-u} + \frac{1}{1-u} \left( \frac{1-u}{e^t-u} \right)^2 \\ &= -\frac{1}{1-u} (F - F^2). \end{aligned}$$

By (2.2), we obtain

$$(2.3) \quad \begin{aligned} F^{(1)}(t, x) &= F^{(1)} e^{xt} = -\frac{1}{1-u} (F(t, x) - F^2(t, x)), \\ (1-u)F^{(1)} + F &= F^2. \end{aligned}$$

**THEOREM 2.1.** For  $u \in \mathbb{C}$  with  $u \neq 1$ ,  $N \in \mathbb{N}$ ,

$F(u) = \frac{1-u}{e^t-u}$  is a solution of

$$(2.4) \quad F^N(u) = \sum_{k=0}^{N-1} \frac{1}{k!} (1-u)^k F^{(k)}(u),$$

where  $F^{(k)}(u) = \frac{d^k F(u)}{du^k}$  and  $F^N(u) = \underbrace{F(u) \times \cdots \times F(u)}_{N\text{-times}}$ .

*Proof.* (Mathematical Induction)

(i) If  $N = 1$ , then it is obvious.

(ii) Assume that (2.4) is true for some  $N > 1$ . Let us consider the derivative of (2.4).

$$(2.5) \quad \begin{aligned} NF^{N-1}F^{(1)} &= \sum_{k=0}^{N-1} \frac{1}{k!} \left( -k(1-u)^{k-1}F^{(k)} + (1-u)^kF^{(k+1)} \right) \\ &= \frac{1}{(N-1)!} (1-u)^{N-1}F^{(N)}. \end{aligned}$$

Thus, from (2.3) and (2.5), we have

$$(2.6) \quad \frac{1}{N!} (1-u)^N F^{(N)} = F^{N-1} (1-u) F^{(1)} = F^{N-1} (-F + F^2)$$

By (2.5) and (2.6), we get

$$F^{N+1} = F^N + \frac{1}{N!} (1-u)^N F^{(N)} = \sum_{k=0}^N \frac{1}{k!} (1-u)^k F^{(k)}.$$

□

**COROLLARY 2.2.** For  $u \in \mathbb{C}$  with  $u \neq 1$ ,  $N \in \mathbb{N}$ ,

$F(t, x) = \frac{1-u}{e^t - u} e^{xt}$  is a solution of

$$(2.7) \quad F^{(N)}(t, x) = \sum_{k=0}^{N-1} \frac{1}{k!} (1-u)^k F^{(k)}(t, x).$$

*Proof.* It is proved from the facts  $F^{(N)}(t, x) = F^{(N)}(u)e^{xt}$  and  $F^{(k)}(t, x) = \frac{d^k F(u)}{du^k} e^{xt}$ . □

### 3. Identities on the Eulerian numbers and polynomials of higher order

**THEOREM 3.1.** For  $N \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ , we have

$$H_n^{(N)}(u) = \sum_{k=0}^{N-1} \frac{1}{k!} (1-u)^k \frac{d^k H_n(u)}{du^k}.$$

*Proof.* By (1.8) and (2.1), we get

$$(3.1) \quad F^N = \underbrace{\frac{1-u}{e^t - u} \times \cdots \times \frac{1-u}{e^t - u}}_{N\text{-times}} = \sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!}.$$

From (1.3) and (2.1), we get

$$(3.2) \quad F^{(k)} = \frac{d^k F(u)}{du^k} = \sum_{n=0}^{\infty} \frac{d^k H_n(u)}{du^k} \frac{t^n}{n!}.$$

By (2.7) and comparing with coefficients of (3.1) and (3.2), we obtain the result of this theorem.  $\square$

COROLLARY 3.2. For  $N \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$ , we have

$$\begin{aligned} & \sum_{l_1+\dots+l_N=n} \binom{n}{l_1, \dots, l_N} H_{l_1}(u)H_{l_2}(u)\cdots H_{l_N}(u) \\ &= \sum_{k=0}^{N-1} \frac{1}{k!} (1-u)^k \frac{d^k H_n(u)}{du^k}. \end{aligned}$$

*Proof.* From (3.1), we obtain

$$\begin{aligned} (3.3) \quad & \sum_{n=0}^{\infty} H_n^{(N)}(u) \frac{t^n}{n!} = \underbrace{\frac{1-u}{e^t-u} \times \dots \times \frac{1-u}{e^t-u}}_{N\text{-times}} \\ &= \left( \sum_{l_1=0}^{\infty} H_{l_1}(u) \frac{t^{l_1}}{l_1!} \right) \times \dots \times \left( \sum_{l_N=0}^{\infty} H_{l_N}(u) \frac{t^{l_N}}{l_N!} \right) \\ &= \sum_{n=0}^{\infty} \left( \sum_{l_1+\dots+l_N=n} \frac{n! H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u)}{l_1! \cdots l_N!} \right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left( \sum_{l_1+\dots+l_N=n} \binom{n}{l_1, \dots, l_N} H_{l_1}(u) H_{l_2}(u) \cdots H_{l_N}(u) \right) \frac{t^n}{n!}. \end{aligned}$$

Therefore, by Theorem 3.1 and (3.3), it is proved.  $\square$

COROLLARY 3.3. For  $N \in \mathbb{N}$ ,  $n \in \mathbb{Z}_+$ , we have

$$H_n^{(N)}(x|u) = \sum_{k=0}^{N-1} \frac{1}{k!} (1-u)^k \sum_{l=0}^n \binom{n}{l} x^{n-l} \frac{d^k H_n(u)}{du^k}.$$

*Proof.* From (1.4) and (3.2), we get

$$\begin{aligned}
 (3.4) \quad F^{(k)}(t, x) &= F^{(k)} e^{xt} \\
 &= \left( \sum_{n=0}^{\infty} \frac{d^k H_n(u) t^n}{du^k n!} \right) \left( \sum_{n=0}^{\infty} \frac{x^n t^n}{n!} \right) \\
 &= \sum_{n=0}^{\infty} \left( \sum_{l=0}^n \binom{n}{l} x^{n-l} \frac{d^k H_l(u)}{du^k} \right) \frac{t^n}{n!}.
 \end{aligned}$$

Therefore, by (1.8), (2.7) and (3.4), it is proved.  $\square$

### References

- [1] L. Carlitz, *Eulerian numbers and polynomials*, Math. Mag. **23** (1959), 247-260.
- [2] L. Carlitz, *The product of two Eulerian polynomials*, Math. Mag. **36** (1963), 37-41.
- [3] J. Choi, D. S. Kim, T. Kim, Y. H. Kim, *A note on some identities of Frobenius-Euler numbers and polynomials*, Int. J. of Math. and Math. Sci., Article ID 861797 (2012), 9 pages.
- [4] J. Choi, T. Kim, Y. H. Kim, *A note on the q-analogue of Euler numbers and polynomials*, Honam Math. J. **33** (2011), no. 4, 529-534.
- [5] J. Choi, T. Kim, Y. H. Kim, B. Lee, *On the (w, q)-Euler numbers and polynomials with weight  $\alpha$* , Proc. Jongjeon Math. Soc. **15** (2012), no. 1, 91-100.
- [6] T. Kim, *Identities involving Frobenius-Euler polynomials arising from non-linear differential equations*, J. of Number Theory **132** (2012), 2854-2865.
- [7] T. Kim, J. Choi, *A note on the product of Frobenius-Euler polynomials arising from the p-adic integral on  $\mathbb{Z}_p$* , Adv. Studies Contemp. Math. **22** (2012), no. 2, 215-223.

\*

Division of General Education-Mathematics  
 Kwangwoon University  
 Seoul 139-701, Republic of Korea  
*E-mail:* jeschoi@kw.ac.kr